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A NOTE ON THE ELASTICA WITH LARGE LOADS†

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1. INTRODUCTION

The problem of the two-dimensional elastica is well known; see, for example [1, 2, 3]. As the load applied to an elastica increases, the elastica takes on a sequence of shapes as shown in Fig. 1a; also, see Fig. 2.30 of [1]. For large values of the load, the resulting shape of the elastica (nelgecting any out-of-plane effects) can be considered as one-half of a loop configuration as shown by the broken lines in Fig. 1a. The loop configuration is often used as a test configuration for the determination of the strength of small diameter fibers (of a few microns in diameter). The test consists of torsionally twisting a fiber one turn and then allowing the ends to move in enough so as to "snap-in" a loop. The ends of the fiber are then pulled apart by increasing load and in so doing the dimensions of the loop decrease; see Fig. 1b. Measurements of the size of the loop then provide information on the maximum strain and stress in the fiber material.

The purpose of this paper is to show how solutions obtained from an analysis of the loop configuration can be carried over to the analysis of the elastica for large values of load. In so doing we obtain a number of simple and interesting analytical results for the elastica and for the loop geometry. In particular, we obtain some simple asymptotic solutions for the shape of the elastica for large loads by deriving a correction to the solution from the loop geometry. Further we present a number of additional results for the elastica which are useful for large deflection analysis. Finally, we note that the method can be used for the large deflection analysis of a cantilever beam with a transverse end load [3–5] and we give the asymptotic results for this case.

2. FORMULATION

We consider an inextensible linearly elastic beam of length L under a load T, Fig. 1a. The differential equation for the deflected shape of the beam can be written in the form [1, 6]

$$\psi'' = -\lambda \sin \psi, \tag{1}$$

where ψ is the tangent angle to the deformed shape, primes are differentiation with respect to the nondimensional distance (s/L), s is the distance along the undeformed beam, λ is the nondimensional load parameter equal to (TL^2/D) and D is the bending stiffness.

The boundary conditions are

$$\psi(0) = 0; \qquad \psi'(1) = 0 \tag{2}$$

and the shape of the deformed beam is determined by integration from

$$x' = \cos \psi; \qquad y' = \sin \psi. \tag{3}$$

Equations (1)–(3) were integrated numerically using a finite-difference boundary value formulation with a Newton-Raphson scheme for the nonlinear correction. The

† Dedicated to Professor Eric Reissner on his Seventieth Birthday.

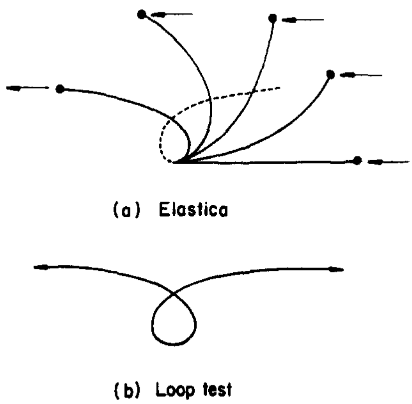

Fig. 1. (a) Geometry of the elastica under increasing load. (b) The loop geometry.

results of the calculations are shown by the solid curves in Figs. 2–4. In these figures we show as a function of load the geometric locations of a number of points along the elastica as well as the end point angle and maximum bending moment. For example, x_n, y_n are the coordinates of the end point of the beam; x_L, y_L are the coordinates of the point on the beam at which $\psi = (\pi/2)$; y_{TL} is the y coordinate of the beam at which $x = 0$; ψ_n is the angle at the end point; and, M_{max} is the maximum bending moment in the beam. A number of cross-plots can be obtained from these curves, for example, M_{max} vs x_L, which would show the maximum moment as a function of a loop diameter for loads greater than 3.6 approximately. The results for ψ_n, x_n and y_n agree with those given in [7, Fig. 46].

From these numerically obtained results for the elastica, we see in Fig. 2 that once the load exceeds 3.6 approximately, the elastica starts to form into one-half of a loop. This suggests that it may be possible to think of the elastica solution for large values of the load in terms of the solution for a loop configuration.

To do this, it is first convenient to rewrite the equations (1)–(3) in terms of a new independent variable $\eta = \sqrt{\lambda}\zeta$. In terms of η, the equations become

$$\psi'' + \sin\psi = 0; \qquad \psi(0) = 0; \qquad \psi'(\sqrt{\lambda}) = 0 \tag{4}$$

where now primes indicate differentiation w.r.t. η.

In this formulation we are seeking the function $\psi(\eta)$ in a domain whose size depends upon the load λ. If $\lambda \to \infty$ then the load does not appear explicitly in the equations nor in the boundary conditions. In this case, the geometry is that of a loop in which $\psi'(\infty) = 0$ with $\psi(\infty) = \pi$; see [8] and a related discussion in [2, Section 1.4].

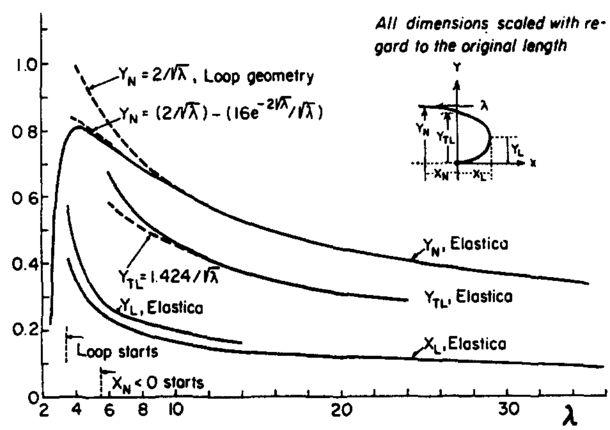

Fig. 2. Positions of points on an elastica, see insert, as a function of applied load λ; solid curves. Approximate solutions, see text, shown by dashed lines.

The solution in this case, with $\lambda \to \infty$, is

$$(\psi/2) = \sin^{-1}(\tanh 2\eta) \tag{5}$$

and the equations for the shape of the beam follow from (3). Upon integration of the y equation we find

$$y = 2[1 - \cos(\psi/2)]/\sqrt{\lambda} \tag{6}$$

from which it follows that as $\psi \to \pi$, $\eta \to \infty$, and

$$y = y_N = 2/\sqrt{\lambda} \tag{7}$$

where y_N is the asymptotic value of the y coordinate as $\eta \to \infty$. When $\psi = (\pi/2)$,

$$y = y_L = 0.5858/\sqrt{\lambda}. \tag{8}$$

Upon integration of the x equation, we find

$$2\sqrt{\lambda}x = \ln\left[\frac{\sqrt{2} - \sqrt{(1 - \cos\psi)}}{\sqrt{2} + \sqrt{(1 - \cos\psi)}}\right] + 2\sqrt{(1 - \cos\psi)}. \tag{9}$$

Again at $\psi = \pi/2$ we find

$$x = x_L = 0.5328/\sqrt{\lambda} \tag{10}$$

and the loop dimension ratio is

$$(y_{TL}/2x_{TL}) = 1.34. \tag{11}$$

These results are in agreement with those obtained in [8, 9].

Finally it follows that $M_{\max} = D\psi'(0) = 2\sqrt{\lambda}$ from which it follows, for example, that

$$M_{\max} = 1.066/x_L = 4/y_N. \tag{12}$$

The results obtained above from the limiting case of a loop geometry agree closely with the corresponding results obtained from the elastica when $\lambda \geq 9$. Figures 2–4

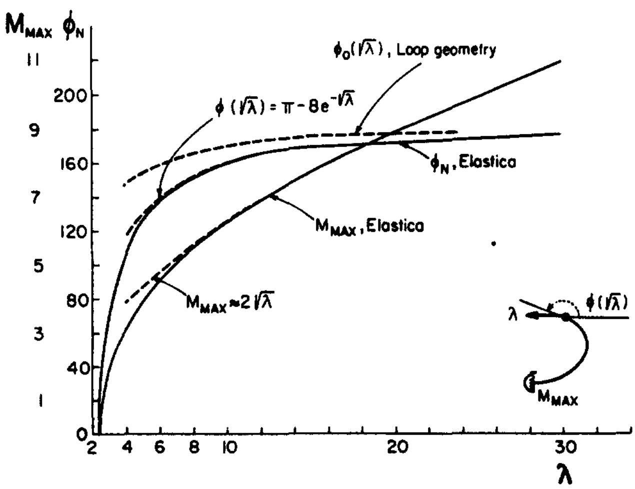

Fig. 3. Angle at end point of the elastica and maximum bending moment as a function of the applied load λ; solid curves. Approximate solutions, see text, shown by dashed lines.

show the above results as dashed curves. The agreement between the results is in fact surprisingly close.

This close agreement suggests that it should be possible to obtain even better agreement by obtaining an asymptotic correction to the results of the loop geometry which will be applicable to the elastica for large values of load.

To do this, we write the solution for ψ in terms of the solution from the loop geometry in the form

$$\phi = \psi - \psi_0; \qquad \psi = \psi_0 + \phi \tag{13}$$

where ψ_0 is the solution for the loop geometry given by (5). The equations for ϕ are

$$\begin{gathered} \psi_0'' + \phi'' + \sin(\psi_0 + \phi) = 0 \\ \phi(0) = 0; \qquad \phi'(\sqrt{\lambda}) = -\psi_0'(\sqrt{\lambda}) \equiv \epsilon. \end{gathered} \tag{14}$$

Since ϵ is small, we look for a solution in an expansion of the form

$$\phi = \epsilon\phi_0 + \epsilon^2\phi_1 + \cdots \tag{15}$$

where we assume $\epsilon\phi_0 \ll 1$, and retain only the first order system:

$$\phi_0'' + \cos\psi_0\phi_0 = 0; \qquad \phi_0(0); \qquad \phi_0'(\sqrt{\lambda}) = 1. \tag{16}$$

We note that the equation for ϕ_0 is a linear system in which $\psi_0 = \psi_0(\eta)$ is known from the loop configuration solution.

It follows that after a little algebra upon rewriting the independent variable as ψ_0 in the differential equation that the solution satisfying the boundary condition at $\eta = 0$, $\psi_0 = 0$ can be written in the form

$$\phi_0 = C[\tan(\psi_0/2) + \eta\cos(\psi_0/2)]. \tag{17}$$

The constant C is determined from the boundary condition $\phi_0'(\sqrt{\lambda}) = 1$ at $\psi_0(\sqrt{\lambda}) = \psi_{0N}$.

Once the solution ϕ_0 is found, corrections to the ψ_0 solution can be found. For example, the angle at $\eta = \sqrt{\lambda}$ is

$$\begin{aligned} \psi(\sqrt{\lambda}) &= \psi_0(\sqrt{\lambda}) + \epsilon\phi_0(\sqrt{\lambda}) + \cdots \\ &= \psi_0(\sqrt{\lambda}) - \psi_0'(\sqrt{\lambda})\phi_0(\sqrt{\lambda}) + \cdots \end{aligned} \tag{18}$$

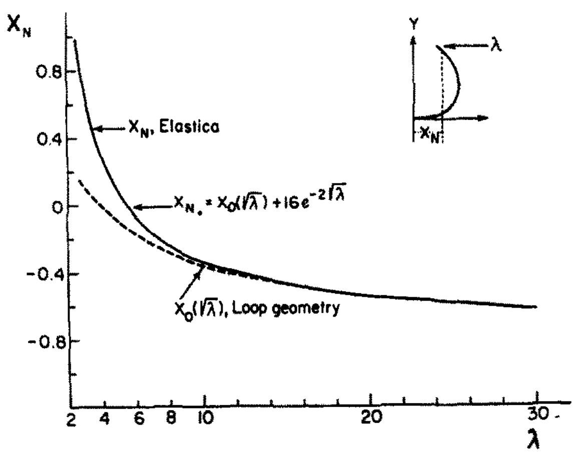

Fig. 4. Displacement in x direction of end point of the elastica as a function of applied load λ; solid curve. Approximate solution shown by dashed curve; eqn (23) agrees closely with elastica solution as shown.

or

$$\psi(\sqrt{\lambda}) = \psi_0(\sqrt{\lambda}) - 2\left[\frac{\sin\psi_0 + 2\sqrt{\lambda}\cos^3(\psi_0/2)}{3 + \cos\psi_0 - \sqrt{\lambda}\sin\psi_0\cos(\psi_0/2)}\right]$$

where

$$\psi_0(\sqrt{\lambda}) = \psi_{0N} = 2\sin^{-1}(\tanh 2\sqrt{\lambda}).$$

A further simplification for large loads is possible if we note that approximately

$$\psi_{0N} = \pi - \epsilon_1(\lambda) = \pi - 4e^{-\sqrt{\lambda}} \tag{19}$$

where ϵ_1 is the deviation from the limiting value of $\psi_{0N} = \pi$ and is a known function of the parameter λ; upon expansion of the solution for ψ_{0N} we obtain the expression for ϵ_1.

It follows that

$$\psi(\sqrt{\lambda}) = \pi - 8e^{-\sqrt{\lambda}}. \tag{20}$$

The corresponding expression for the tip displacement is

$$\sqrt{\lambda}\, y_N = 2 - 16e^{-2\sqrt{\lambda}}. \tag{21}$$

Finally the expression of x_N can be obtained in the form

$$x_N = x_{0N} - (\epsilon/\sqrt{\lambda})C\left[\frac{3\eta}{2} - \frac{3}{4}\sin(\psi_0/2) + \frac{\eta}{2}\sin^2(\psi_0/2)\right]_{\eta=\sqrt{\lambda}}. \tag{22}$$

An approximate expression for x_N follows upon expansion in the form

$$x_N = x_{0N} + 16e^{-2\sqrt{\lambda}}. \tag{23}$$

The results for x_N, y_N, and $\psi(\sqrt{\lambda})$ given in (23), (21), (20) are plotted in Figs. 2–4 and the agreement with the numerical solution is very good down to loads as small as $\lambda = 4$. In fact the approximation for x_N is in close agreement down to $\lambda = 3$; see Fig. 4.

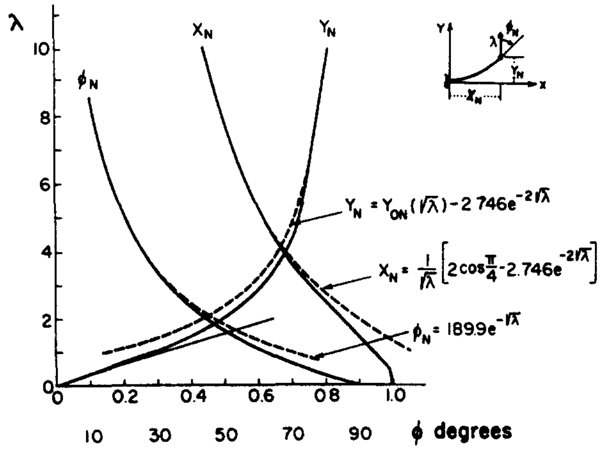

Fig. 5. Solutions for end point displacements and end point angle of a transversely loaded cantilever beam from Ref. [2, 4], solid curves, and approximate analytical solutions shown by the dashed curves.

Finally if we consider the problem of the cantilever beam loading with a transverse load,

$$\psi'' = -\lambda \cos \psi$$

we find that the analysis for the elastica can be used if we let $\psi \to \psi + \pi/2$. We do not show the details here but the results are shown in Figure 5 which shows the curves of Bisshopq and Drucker [4] and those from [2, Fig. 2.4] and the approximate asymptotic results. Again the results are in close agreement.

3. CONCLUSIONS

A number of simple results for the elastica and for the cantilever beam for large values of applied load have been obtained. The method used was to correct by means of an asymptotic expansion the solution for the loop geometry which can be considered as the limiting case of an infinite load for the elastica. The results should be useful for the analysis of large deflections of very flexible beams and filaments.

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